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Classical and quantum dynamics in a random magnetic field

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Abstract. Using the supersymmetry approach, we study spectral statistical properties of a two-dimensional quantum particle subject to a non-uniform magnetic field. We focus mainly on the problem of regularization of the field theory. Our analysis begins with an investigation of the spectral properties of the purely classical evolution operator. We show that, although the kinetic equation is formally time-reversible, density relaxation is controlled by *irreversible* classical dynamics. In the case of a weak magnetic field, the effective kinetic operator corresponds to diffusion in the angle space, the diffusion constant being determined by the spectral resolution of the inhomogeneous magnetic field. Applying these results to the quantum problem, we demonstrate that the low-lying modes of the field theory are related to the eigenmodes of the irreversible classical dynamics, and the higher modes are separated from the zero mode by a gap associated with the lowest density relaxation rate. As a consequence, we find that the long-time properties of the system are characterized by universal Wigner–Dyson statistics. For a weak magnetic field, we obtain a description in terms of the quasi one-dimensional nonlinear σ-model.

1. Introduction

Over recent years, the phenomenon of quantum chaos has been the subject of intense theoretical and experimental investigations [1]. Of these, the most recent has been the development of a statistical field theory in which the quantum properties of classically chaotic systems are expressed through the modes of density relaxation specified by the classical evolution operator [2]. In this paper, we apply this field theoretic procedure to investigate the quantum dynamics of a two-dimensional particle subject to a random magnetic field (RMF). Our choice is motivated by two factors. First, this problem represents one of the simplest examples of a chaotic system. Secondly, it is of great practical interest, and as such has already received a lot of attention in the literature [3–17]. To put this investigation in perspective, we first describe the basis of the general field theoretic construction. Then, we review the main results on the RMF problem, and outline the strategy for the rest of the paper.

The field theoretic approach to general chaotic quantum structures has been motivated by the success enjoyed by the statistical field theory of weakly disordered metallic conductors. More specifically, applied to a model in which time-reversal symmetry is broken, an average over realizations of a Gaussian δ-correlated random impurity potential shows that the two-particle spectral properties of a weakly disordered Hamiltonian are described by a nonlinear σ-model with the effective action [18]

$$S = -\frac{\pi\nu}{4}\int \mathrm{d}\boldsymbol{r}\ \mathrm{str}[\hbar D(\nabla Q)^2 + 2\mathrm{i}s^+\Delta\sigma_3^{\mathrm{AR}}Q] \qquad (1)$$

where the 4×4 supermatrix fields $Q(\boldsymbol{r})$, which exhibit a hierarchical structure composed of advanced–retarded (AR) and boson–fermion (BF) blocks, are subject to the nonlinear constraint

$Q^2(\mathbf{r}) = \mathbb{1}$. Here $\nu = 1/V\Delta$ represents the average density of states per unit volume, Δ is the mean level spacing, D denotes the diffusion constant, $s^+ \equiv s + \mathrm{i}0$ represents the energy source, $\sigma_3^{\mathrm{AR}} = \mathbb{1}_{\mathrm{BF}} \otimes \mathrm{diag}(1, -1)_{\mathrm{AR}}$ is the symmetry breaking matrix in the AR sector, and str denotes the trace operation for supermatrices, $\mathrm{str}\ M = \mathrm{tr}\ M_{\mathrm{FF}} - \mathrm{tr}\ M_{\mathrm{BB}}$. With this definition, disorder-averaged quantities such as $\langle G^{\mathrm{A}}(E) G^{\mathrm{R}}(E + s\Delta)\rangle$, where indices A and R refer to the advanced and retarded Green functions, can be calculated as field correlators of the model (1). Formally, the supermatrices Q belong to the coset space $U(1, 1|2)/U(1|1) \otimes U(1|1)$. Parametrized in the form $Q(\mathbf{r}) = T^{-1}(\mathbf{r})\sigma_3^{\mathrm{AR}} T(\mathbf{r})$, the generators T can be identified with soft diffusion modes of density relaxation from which mechanisms of quantum interference can be described. (An extensive review of this standard formalism can be found in [18].)

The connection between the supersymmeric field theory and physical coherence effects in disordered conductors can be understood by analysing characteristic scales of the theory. The effective action (1) identifies two timescales, the diffusion time $t_D = L^2/D$ and the Heisenberg time $t_H = \hbar/\Delta$. Their ratio defines the dimensionless conductance which in a good metal is large: $g \equiv t_H/t_D \gg 1$. For energies $s\Delta \ll \hbar/t_D$, where quasi-classical dynamics is *ergodic*, the effective action is dominated by the zero spatial mode, $Q(\mathbf{r}) = Q_0$, independent of $\mathbf{r}$

$$S[Q_0] = -\mathrm{i}\frac{\pi s^+}{2}\,\mathrm{str}[\sigma_3^{\mathrm{AR}} Q_0]. \tag{2}$$

In this limit, spectral properties are universal and, in this case, coincide with those of random matrix ensembles of unitary symmetry. Conversely, on energy scales $s\Delta \gg \hbar/t_D$, the effective action is dominated by the global saddle-point $Q_{\mathrm{sp}} = \sigma_3^{\mathrm{AR}}$. An expansion in terms of the generators of the coset identifies the low-lying modes of density relaxation as diffusion modes. Interaction of these modes induces the well known quantum weak localization corrections.

Surprisingly, at least phenomenologically, an analogous theory can be constructed for systems which are non-integrable but not stochastic. Guided by the quasi-classical Boltzmann description of density relaxation in ballistic transport, in an insightful work by Muzykantskii and Khmel'nitskii [2], a generalization of the diffusive action (1) was proposed in which relaxation is controlled by the kinetic operator. The corresponding 'ballistic' action takes the *general* form

$$S[\mathcal{Q}] = -\mathrm{i}\frac{\pi}{2}\int \mathrm{d}\boldsymbol{x}_\parallel\ \mathrm{str}[s^+\Delta\sigma_3^{\mathrm{AR}}\mathcal{Q} + 2\mathrm{i}\hbar\mathcal{T}^{-1}\sigma_3^{\mathrm{AR}}\hat{\mathcal{L}}\mathcal{T}] \tag{3}$$

where, as before, $\mathcal{Q}(\boldsymbol{x}_\parallel) = \mathcal{T}^{-1}(\boldsymbol{x}_\parallel)\sigma_3^{\mathrm{AR}}\mathcal{T}(\boldsymbol{x}_\parallel)$ represents a 4×4 supermatrix obeying the nonlinear constraint $\mathcal{Q}^2(\boldsymbol{x}_\parallel) = \mathbb{1}$. (Symmetry properties of matrices $\mathcal{Q}$ and $\mathcal{T}$ are discussed in detail in [19]). Again, the zero-mode contribution to the effective action (3) reproduces random matrix or Wigner–Dyson statistics [20], while the higher mode fluctuations establish non-universal quantum corrections.

Along with apparent similarities, there are also important differences between actions (3) and (1): first, the matrix fields $\mathcal{Q}$ depend on the $2d-1$ *phase space* coordinates $\boldsymbol{x}_\parallel = (\mathbf{r}, \mathbf{p})_{2d-1}$ which parametrize the constant energy shell $H(\mathbf{r}, \mathbf{p}) = E$. (Here the coordinates are normalized such that $\Delta\int \mathrm{d}\boldsymbol{x}_\parallel = 1$.) Secondly, the diffusion operator in equation (1) is now replaced by the kinetic operator, the classical Poisson bracket, $\hat{\mathcal{L}} = \{H, \ \}$. Accordingly, equation (3) offers a description of quantum chaotic systems in terms of the classical dynamics in the phase space.

After its introduction (within the framework of ballistic transport), the phenomenology of Muzykantskii and Khmel'nitskii subsequently found support in the work of Andreev *et al* [21]. Specifically, recognizing that the spectrum of an individual system provides a statistical ensemble over which an average can be defined, a more direct construction of the effective action provided further credence to the phenomenology. Yet important technical problems continue to confound a truly rigorous derivation of the ballistic action. (See [22] for a discussion.)

The utility of the field-theoretical approach beyond the universal regime has been demonstrated in papers by Blanter *et al* [23], and Tripathi and Khmel'nitskii [24] where the effective action (3) was applied to the study of a quantum billiard with diffusive surface scattering, a system that exemplifies a ballistic system in the regime of strong chaos. However, the σ-model approach has so far failed to provide explicit results for truly clean systems. This can, in part, be attributed to the fact that the kinetic operator $\hat{\mathcal{L}}$, which enters the effective action, is antiHermitian for any ballistic system, a signature of the reversible nature of classical dynamics. As a consequence, its eigenvalues lie on the imaginary axis, so that the time evolution of a distribution in the phase space does not exhibit relaxation into the uniform ergodic state. Moreover, for chaotic systems, $\hat{\mathcal{L}}$ is ill-defined in the following sense: classical dynamics involves stretching along the unstable manifold and contraction along the stable one. Thus, any non-uniform initial distribution will evolve into a highly singular function. In terms of the field theory, it means that the functional integral suffers from ultraviolet divergencies. In order to understand this, let us notice that the action (3) is only sensitive to the variations of the $\mathcal{Q}$-matrix along the classical trajectories. Therefore, nothing prevents the $\mathcal{Q}$-matrix from fluctuating in the transverse direction. It is these short-scale fluctuations that ultimately lead to the divergence of the functional integral [21].

Several regularization procedures have been proposed to circumvent this problem [25–28]. One of the most natural ways to regularize the functional integral is to introduce an additional term into the effective action which suppresses the fluctuations of $\mathcal{Q}$ in the transverse direction. After performing the integration, one should take the regulator to zero. A surprising feature of the chaotic dynamics is that the limits time-to-infinity and regulator-to-zero do not commute. In particular, the eigenvalues of the regularized kinetic operator in the limit regulator-to-zero remain complex, with finite real parts corresponding to relaxation rates into the equilibrium distribution. These complex eigenvalues, which are independent of the regularization procedure, reflect intrinsic irreversible properties of the classical chaotic dynamics, and are known as Ruelle resonances or the Perron–Frobenius spectrum [29, 30].

One should stress that the direct regularization method described above is of little practical use. This has been the main reason why all the attempts to apply the ballistic σ-model to real systems have been unsuccessful. In this paper we propose a new general approach to the regularization problem, which allows one to construct an effective field theory for the low-lying part of the Perron–Frobenius spectrum, and is applicable for both classical and quantum systems. To describe the basis of this approach, let us recall that the kinetic operator is defined in some Hilbert space, elements of which are smooth functions of the phase space coordinates. As was mentioned earlier, the eigenfunctions of $\hat{\mathcal{L}}$ are highly singular and lie outside the Hilbert space. However, by properly choosing a subspace of the full Hilbert space one can make $\hat{\mathcal{L}}$ a well-defined operator in the sense that its eigenfunctions belong to the same subspace. Obviously, this subspace should correspond to the physically relevant low-lying part of the Perron–Frobenius spectrum. As we will see later, the kinetic operator defined in such a way is irreversible. To summarize, instead of calculating the full spectrum of the regularized kinetic operator, one can instead obtain an effective operator that correctly describes the low-lying modes of the irreversible classical dynamics.

In order to choose the subspace in which the effective kinetic operator will act, one should truncate the basis in the Hilbert space by eliminating degrees of freedom irrelevant for the long-time evolution. Operationally, this is done best in the field-theoretical formulation. To illustrate our approach, let us consider the following formalism, wherein the Green function of the classical kinetic operator is represented as an integral over the superfield $\psi = (\psi_{\rm B}, \psi_{\rm F})$,

$$\hat{g}(\omega) \equiv \frac{1}{{\rm i}\omega - \hat{\mathcal{L}}} = \int {\rm D}\psi^* \, {\rm D}\psi \; \psi_{\rm B} \otimes \psi_{\rm B}^* \exp\left[- \int {\rm d}\boldsymbol{x}_\parallel \, \psi^\dagger(\boldsymbol{x}_\parallel)({\rm i}\omega - \hat{\mathcal{L}})\psi(\boldsymbol{x}_\parallel)\right]. \tag{4}$$

We then employ an RG-type approach and decompose ψ in the following way,

$$\psi(\boldsymbol{x}_\parallel) = \Psi(\boldsymbol{x}_\parallel) + \chi(\boldsymbol{x}_\parallel) \tag{5}$$

where $\Psi(\boldsymbol{x}_\parallel)$ denotes 'slow' fields, and $\chi(\boldsymbol{x}_\parallel)$ 'fast' fields. Integrating over the 'fast' fields, χ, one obtains an expression for the effective Green function in the Ψ-subspace,

$$\hat{g}(\omega) = \int {\rm D}\Psi^* \, {\rm D}\Psi \; \Psi_{\rm B} \otimes \Psi_{\rm B}^* \exp\left[- \int {\rm d}\boldsymbol{x}_\parallel \, \Psi^\dagger(\boldsymbol{x}_\parallel)({\rm i}\omega - \hat{\mathcal{L}}_{\rm eff})\Psi(\boldsymbol{x}_\parallel)\right] \tag{6}$$

containing a renormalized kinetic operator $\hat{\mathcal{L}}_{\rm eff}$. There is no universal recipe as to how one should identify the 'fast' and 'slow' fields. However, this can be done straightforwardly in the case of 'weak non-integrability'†, if the kinetic operator can be represented in the form

$$\hat{\mathcal{L}} = \hat{\mathcal{L}}_0 + \hat{\mathcal{L}}_{\rm pert} \tag{7}$$

where $\hat{\mathcal{L}}_0$ describes an integrable classical system, and $\hat{\mathcal{L}}_{\rm pert}$ can be considered as a small perturbation. In this case one can, in principle, calculate the eigenfunctions and eigenfrequencies of the unperturbed operator $\hat{\mathcal{L}}_0$. 'Fast' fields are identified as eigenfunctions of $\hat{\mathcal{L}}_0$ corresponding to high frequencies, while 'slow' fields correspond to the low-frequency part of the spectrum of $\hat{\mathcal{L}}_0$.

The main purpose of the paper is to demonstrate how this general regularization procedure works by applying it to the problem of a particle confined to two dimensions and propagating in a random magnetic field. The latter has attracted great interest in recent years. First, there exist a number of experimental realizations where a RMF is imposed on a two-dimensional electron gas. These include cases where the RMF is imposed by randomly pinned flux vortices in a type-II superconducting gate [3], by grains of a type-I superconductor [4], or by a demagnetized permanent magnet placed in the vicinity of the electron gas [5]. Secondly, models of this type have been proposed within the gauge theory of high-T_c superconductivity [6, 7]. Thirdly, the problem is thought to be relevant for the composite fermion theory of the quantum Hall effect near $\nu = \frac{1}{2}$ [8, 9].

The classical description of the transport properties of two-dimensional charged particles in a random magnetic field with *long-range* correlations can be given in terms of the distribution function $f(\boldsymbol{r}, \boldsymbol{p})$, which obeys the Boltzmann equation

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\frac{\partial f}{\partial \boldsymbol{r}} + \frac{e}{c}[\boldsymbol{v}, \boldsymbol{B}]\frac{\partial f}{\partial \boldsymbol{p}} = -\frac{f - f_0}{\tau} \tag{8}$$

where the right-hand side describes relaxation to the uniform distribution $f_0 = \delta(v - v_F)$ due to impurity potential scattering. Assuming the magnetic field is weak, one can calculate the averaged Green function of equation (8) using a perturbation theory. Within the Born approximation the renormalized value of the transport scattering time τ_{tr} can be found [10]. In particular, in the absence of the potential scattering ($\tau \to \infty$), one obtains for the transport time [11]

$$\frac{1}{\tau_{tr}} = \frac{1}{mp_F}\int_0^\infty {\rm d}r \, \langle B(\boldsymbol{0})B(\boldsymbol{r})\rangle \tag{9}$$

where the angle brackets denote the ensemble average.

† To eliminate possible confusion, we stress that the system is assumed to be fully ergodic throughout the paper; in other words, we are in the regime of hard chaos, and the KAM theorem is not applicable here.

In the opposite case of a strong magnetic field, where the Born approximation is not applicable, a number of other approaches have been developed for solving the Boltzmann equation, including the 'Eikonal' approximation [12], and the percolation theory [11]. Another possible approach is to use known results from the time-independent diffusion–advection or Passive scalar equation

$$\frac{\partial n}{\partial t} = \mathrm{D}\nabla^2 n + \boldsymbol{u} \cdot \nabla n \tag{10}$$

where $\boldsymbol{u}$ is a random velocity field. Indeed, starting with equation (8) and assuming a rapid relaxation in the momentum space, one obtains an effective equation for the particle density $n(\boldsymbol{r}) = \int \mathrm{d}\boldsymbol{p}\, f(\boldsymbol{r}, \boldsymbol{p})$, which has the form of equation (10). It can be interpreted as the advection of guiding centres (or 'van Alfven drift' of cyclotron orbits) due to the random component of the magnetic field. The diffusion–advection problem has been extensively studied because of its importance in fluid dynamics and plasma physics [13]. Known results suggest a non-trivial scaling behaviour of the conductivity σ_{xx}, which has been confirmed in recent experiments [14]. It is also interesting to note that an analogy exists between the diffusion–advection problem and the localization problem for a quantum particle in a random vector potential [15].

The quantum transport properties of a particle subject to a random magnetic field have been studied by Aronov *et al* [16]. Applying a Gaussian δ-correlated distribution for the magnetic field, spectral and transport properties are shown to be described by a diffusive supersymmetric nonlinear σ-model of unitary symmetry. Although the relaxation rate which enters the average single-particle Green function is divergent if calculated perturbatively, the transport relaxation time, which is a characteristic of the average two-particle Green function, turns out to be finite. The question of finding a physically meaningful definition of the single-particle relaxation time has been discussed by Altshuler *et al* [17].

In contrast to the above work by Aronov *et al*, we want to focus on the case of a weak slowly varying magnetic field. It is convenient for our purposes to divide the rest of the paper into two sections. In the first section, we deal with the classical problem. The procedure of separating slow and fast fields, outlined above, is carried out in detail, and the effective classical kinetic operator is evaluated. The latter is shown to contain a term describing diffusion in the angle space. We calculate the corresponding 'diffusion constant' (transport time), which is determined by the power spectrum of the magnetic field. The angle diffusion term is responsible for the relaxation of the momentum-dependent degrees of freedom, leading to the conventional diffusive dynamics at large scales.

In the second section, we consider the quantum problem, and apply the quasi-classical field theoretic description (3). We show that, cast in the form of a field theory, the classical and quantum problems are closely related. One can therefore separate slow and fast degrees of freedom in equation (3) in almost the same way as one does for the classical problem. The only complication is the somewhat more complex structure of the integration manifold. Taking this into account, and using results from the first section, we arrive at the following conclusions: at short scales (small system size), spectral properties are described by a quasi-one-dimensional σ-model. At intermediate scales, we obtain a ballistic σ-model, where the traditional collision integral is replaced with diffusion in the angle space. At very large scales, we recover the conventional diffusive σ-model. In this way, we demonstrate that a description in terms of the diffusive σ-model holds not only for systems with δ-correlated magnetic fields dominated by quantum scattering [16], but also for semi-classical motion in a long-range magnetic field. In other words, although the semi-classical description is formally valid, quantum corrections are essential, and the transport at large scales is dominated by localization effects. Finally, it is important to emphasize that, in contrast to the existing literature on this problem, we do not make use of the notion of an ensemble average, so that our results are valid for an *individual* system with a given configuration of the magnetic field.

2. Classical dynamics

As mentioned above, as a precursor to the analysis of the quantum problem, we begin by considering the dynamics of a *classical* particle in two-dimensions subject to an inhomogeneous perpendicular magnetic field. Indeed, provided that $\hbar/mv \ll a$, where m is the mass of the particle, v is its velocity, and a is the typical length scale at which the magnetic field fluctuates, the motion of the quantum particle can be described semi-classically. Since energy is conserved, $|\boldsymbol{v}| = \text{const.}$, the motion is defined by the set of phase space variables (x, y, ϕ), which obey the following equations

$$\dot{x} = v\cos\phi \qquad \dot{y} = v\sin\phi \qquad \dot{\phi} = \Omega(x, y) \tag{11}$$

where $\Omega = eB/mc$ represents the cyclotron frequency. We will assume that $\Omega(x, y)$ is a random function of the spatial coordinates with zero average. We will further assume that the magnetic field is weak: $\Omega a \ll v$.

As the particle moves through areas of different magnetic field, its trajectory is slightly deflected to one side or another depending on the sign of the magnetic field. From the point of view of the particle, the magnetic field $\Omega(x(t), y(t))$ is a random function of time. As such, one can obtain a good qualitative description of the particle's motion by taking this function to be δ-correlated Gaussian noise:

$$\begin{aligned} &\dot{\phi} = \Omega(t) \\ &\langle \Omega(t)\Omega(t')\rangle = \frac{2}{\tau_{tr}}\delta(t - t'). \end{aligned} \tag{12}$$

From dimensional considerations it is clear that $1/\tau_{tr} \sim \Omega^2 a/v$. From equation (12) one obtains

$$\langle \phi^2(t)\rangle = \int_0^t \int_0^t \mathrm{d}t'\,\mathrm{d}t''\,\langle \Omega(t')\Omega(t'')\rangle = \frac{2}{\tau_{tr}}t. \tag{13}$$

Such time dependence of $\langle \phi^2\rangle$ corresponds to diffusion in the angle space, with the diffusion coefficient $1/\tau_{tr}$. If $t \lesssim a/v$, the motion is ballistic, and our description fails. At times $a/v \ll t \ll \tau_{tr}$ (the existence of this intermediate regime is ensured by the condition of weakness of the magnetic field $\Omega a \ll v$) one has $\langle \phi^2\rangle \ll 1$, so that the particle's trajectory in real space remains an approximately straight line. Finally, at times $t \gtrsim \tau_{tr}$, angle diffusion results in the uniform distribution of angles. Physically, this means that the direction of movement becomes random, which leads to diffusion in real space (diffusive regime). Indeed, one has

$$\langle x^2(t)\rangle = v^2 \int_0^t \int_0^t \mathrm{d}t'\,\mathrm{d}t''\,\langle \cos\phi(t')\cos\phi(t'')\rangle \approx 2Dt \tag{14}$$

where

$$D = \frac{v^2\tau_{tr}}{2} \tag{15}$$

represents the diffusion coefficient in real space.

This description offers a clear physical picture of what is going on in the system. However, it has some drawbacks. First, since δ-correlation contradicts the existence of a finite correlation length a of the magnetic field, it is not self-consistent. Secondly, a and Ω themselves are not well-defined quantities. After all, this is just a phenomenological description. We would like to obtain a quantitative description that would relate the diffusion coefficient to specific spectral properties of the random function $\Omega(x, y)$. In order to do this we make use of an alternative representation of the problem in the form of the Boltzmann equation. This representation has the advantage of being linear, allowing us to apply a field-theoretical approach.

We start by recalling some basic facts about the Boltzmann equation: any classical system obeying Hamilton's equations of motion can be alternatively described in terms of the distribution function $f(\boldsymbol{x})$, which obeys the following partial differential equation

$$\frac{\partial f}{\partial t} + \hat{\mathcal{L}} f = 0 \tag{16}$$

where the kinetic operator $\hat{\mathcal{L}}$ describes an incompressible flow in the phase space,

$$\hat{\mathcal{L}} = \dot{\boldsymbol{x}} \cdot \frac{\partial}{\partial \boldsymbol{x}} \qquad \partial \cdot \dot{\boldsymbol{x}} = 0. \tag{17}$$

The incompressibility of the flow has a number of important implications, including the conservation of particle number, and the antihermiticity of $\hat{\mathcal{L}}$. In particular, for two arbitrary states $|\psi\rangle$ and $|\chi\rangle$, the following relation holds:

$$\langle \psi | \hat{\mathcal{L}} | \chi \rangle^* = -\langle \chi | \hat{\mathcal{L}} | \psi \rangle. \tag{18}$$

Consequently, the eigenvalues of $\hat{\mathcal{L}}$ are purely imaginary numbers. In the introduction we discussed the reasons why the spectrum of the kinetic operator for a chaotic system is not well defined, and came to the conclusion that the kinetic operator should be regularized. In the simplest case of a free particle, this can be done by introducing a term which accounts for the relaxation of the fast modes into the right-hand side of equation (16):

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} = u \partial^2 f \qquad u > 0. \tag{19}$$

Substituting $f = f_{\boldsymbol{k},\omega} \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r} - \mathrm{i}\omega t)$ into equation (19) and taking the limit $u \to 0$, one finds that the eigenvalues $\mathrm{i}\omega(\boldsymbol{k})$ of $\hat{\mathcal{L}}$ acquire an infinitesimal positive real part:

$$\omega(\boldsymbol{k}) = \boldsymbol{v} \cdot \boldsymbol{k} - \mathrm{i}0. \tag{20}$$

As suggested in the introduction, the eigenvalues of a generic Perron–Frobenius operator have *finite* positive real parts.

For the problem at hand, it is convenient to introduce the parametrization

$$\boldsymbol{v} = v\boldsymbol{n}(\phi) \qquad \boldsymbol{n}(\phi) = (\cos\phi, \sin\phi). \tag{21}$$

The kinetic energy of a particle in the magnetic field is conserved, so that the phase space $\boldsymbol{x} = (\boldsymbol{r}, \boldsymbol{p})$ is effectively reduced to the energy shell $\boldsymbol{x}_\parallel = (\boldsymbol{r}, \phi)$, and the kinetic operator takes the form

$$\hat{\mathcal{L}} = v\boldsymbol{n}(\phi) \cdot \frac{\partial}{\partial \boldsymbol{r}} + \Omega(\boldsymbol{r}) \frac{\partial}{\partial \phi}. \tag{22}$$

Since $\hat{\mathcal{L}}$ is a linear operator, equation (16) can be solved. The general solution can be written in the form

$$f(\boldsymbol{x}, t) = \hat{U}(t) f(\boldsymbol{x}, 0) \tag{23}$$

where $\hat{U}(t) = \exp(-\hat{\mathcal{L}}t)$ represents the classical evolution operator. It is often more convenient to work with its Fourier transform

$$\hat{g}(\omega) = -\int_0^\infty \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} \hat{U}(t) = \frac{1}{\mathrm{i}\omega - \hat{\mathcal{L}}}. \tag{24}$$

If we had an ensemble of different configurations of the magnetic field, information about transport properties could be extracted from the ensemble averaged distribution function

$\langle f(\boldsymbol{x}, t)\rangle$. As follows from equations (23) and (24), it would be sufficient to calculate the ensemble averaged evolution operator $\langle \hat{U}(t)\rangle$ or the Green function $\langle \hat{g}(\omega)\rangle$. One could then introduce the effective 'ensemble averaged' kinetic operator $\hat{\mathcal{L}}_{\rm eff}$ in the following manner:

$$\langle \hat{U}(t)\rangle = \exp(-\hat{\mathcal{L}}_{\rm eff} t). \tag{25}$$

Since $\hat{\mathcal{L}}$ is antiHermitian, $\hat{U}(t)$ is a unitary operator. The ensemble averaged unitary operator, however, is not necessarily unitary, which means that $\hat{\mathcal{L}}_{\rm eff}$ is *not* antiHermitian, so that the spectrum of $\hat{\mathcal{L}}_{\rm eff}$ has a real part. Completely reversible dynamics becomes irreversible after the averaging.

One can, in principle, derive the spectrum of relaxation modes by calculating $\langle \hat{g}(\omega)\rangle$. But, since we are dealing with an individual system, not with an ensemble of systems, we are bound to take a different route. Following the ideas explained in the introduction, we therefore represent the classical Green function (24) as a field integral,

$$\hat{g}(\omega) = \int {\rm D}\psi^* \, {\rm D}\psi \; \psi_{\rm B} \otimes \psi_{\rm B}^* \exp\{-S[\psi^*, \psi]\} \tag{26}$$

where

$$S = \int {\rm d}\boldsymbol{r} \, {\rm d}\phi \; \psi^\dagger ({\rm i}\omega - \hat{\mathcal{L}})\psi \tag{27}$$

denotes the effective action, and $\psi(\boldsymbol{r}, \phi) = (\psi_{\rm B}, \psi_{\rm F})$ represents a two-component supervector field, with subscripts B and F corresponding to bosonic and fermionic components. To make the functional integral (26) formally convergent, one usually puts an infinitesimal regulator into the action (27). This is essentially the same regulator as in equation (20), and it will play an important role in the following analysis.

The first term of the operator (22) describes the evolution of a free particle. Note that any function $\Psi(\phi)$ which depends on ϕ but not on $\boldsymbol{r}$ is necessarily a zero-mode of $\hat{\mathcal{L}}_{\rm free} \equiv v\boldsymbol{n}(\phi) \cdot \partial/\partial \boldsymbol{r}$ (i.e. $\hat{\mathcal{L}}_{\rm free}\Psi(\phi) = 0$). This implies that the angle distribution (in ϕ-space) remains constant in time. Switching on a weak magnetic field, the angle distribution starts changing slowly. The evolution in ϕ-space is described by some effective operator which, following on from our naive physical picture, must have a diffusive form with some diffusion coefficient $1/\tau_{tr}$. As we will see later, $1/\tau_{tr}$ is determined by spectral properties of $\Omega(\boldsymbol{r})$.

This observation suggests a separation into slow and fast degrees of freedom from which the fast degrees of freedom can be integrated away. Since the particle's velocity is high and the magnetic field is weak, the spatial coordinate $\boldsymbol{r}(t)$ of the particle changes rapidly, while $\phi(t)$ changes slowly. The same statements are true for the distributions in the $\boldsymbol{r}$-space and ϕ-space respectively. In terms of the action, one could say that spatially-dependent modes are fast, and spatially-independent modes are slow. The corresponding timescales are $\tau_c = L/v$ and τ_{tr}, where L is the size of the system. One can always choose L such that $\tau_c \ll \tau_{tr}$, and at the same time $L \gg a$, the consistency of the two inequalities being ensured by the weakness of the magnetic field. (This is true only if the correlation length a of the magnetic field is finite—see below.) In this case, one can separate the modes with $\boldsymbol{k} = 0$ and perform the integration over the rest of the modes. By doing so, one obtains an effective action describing the evolution in the ϕ-space. For larger systems, a spatial dependence becomes important. The latter can be accounted for by dividing the system into smaller sub-systems and repeating the above procedure for each sub-system. Finally, in the case of very large systems, $\tau_c \gg \tau_{tr}$, the effective action is dominated by spatial diffusion modes.

Beginning with small systems, we adopt the representation

$$\psi(\boldsymbol{r}, \phi) = \frac{1}{\sqrt{V}}\Psi(\phi) + \frac{1}{\sqrt{V}}\sum_{k\neq 0} \chi_{\boldsymbol{k}}(\phi){\rm e}^{{\rm i}\boldsymbol{k}\cdot\boldsymbol{r}} \qquad \Omega(\boldsymbol{r}) = \frac{1}{\sqrt{V}}\sum_{k\neq 0}\Omega_{\boldsymbol{k}}{\rm e}^{{\rm i}\boldsymbol{k}\cdot\boldsymbol{r}} \tag{28}$$

where $\chi_{\boldsymbol{k}}(\phi)$ are small fluctuating fields, and V denotes the volume of the system. Substituting equation (28) into (27), we obtain

$$S = \int d\phi \left\{ i\omega\Psi^\dagger\Psi - \sum_{k\neq 0} \left[iv\boldsymbol{k}\cdot\boldsymbol{n}(\phi)\chi_{\boldsymbol{k}}^\dagger\chi_{\boldsymbol{k}} + \frac{1}{\sqrt{V}}\chi_{\boldsymbol{k}}^\dagger\Omega_{\boldsymbol{k}}\frac{\partial\Psi}{\partial\phi} - \frac{1}{\sqrt{V}}\Omega_{\boldsymbol{k}}^*\frac{\partial\Psi^\dagger}{\partial\phi}\chi_{\boldsymbol{k}} \right] \right\} \tag{29}$$

where we have kept terms of up to second order. Performing the Gaussian integration over $\chi_{\boldsymbol{k}}(\phi)$, we obtain

$$\hat{g}(\omega) = \frac{1}{V}\int D\Psi^* \, D\Psi \; \Psi_B \otimes \Psi_B^* \exp\{-S_{\text{eff}}[\Psi^*, \Psi]\} \tag{30}$$

where

$$S_{\text{eff}} = \int d\phi \left\{ i\omega\Psi^\dagger\Psi - \frac{1}{\tau_{tr}}\frac{\partial\Psi^\dagger}{\partial\phi}\frac{\partial\Psi}{\partial\phi} \right\} \tag{31}$$

denotes the effective action, and

$$\frac{1}{\tau_{tr}} = \frac{1}{V}\sum_{\boldsymbol{k}}\frac{|\Omega_{\boldsymbol{k}}|^2}{i\boldsymbol{v}\cdot\boldsymbol{k}}. \tag{32}$$

(Note that consistency requires only the zeroth-order term to be kept in the pre-exponential.) Replacing the summation in equation (32) by integration, and introducing the infinitesimal regulator according to equation (20), one finds

$$\frac{1}{\tau_{tr}} = -i\int \frac{|\Omega_{\boldsymbol{k}}|^2 \, d\boldsymbol{k}}{\boldsymbol{v}\cdot\boldsymbol{k} - i0} = \frac{2\pi}{v}\int_0^\infty dk\, |\Omega_k|^2 \tag{33}$$

assuming that the spectrum of $\Omega(\boldsymbol{r})$ is isotropic, i.e. depends only on $|\boldsymbol{k}|$. The effective action (31) corresponds to the kinetic operator of the form

$$\hat{\mathcal{L}}_{\text{eff}} = -\frac{1}{\tau_{tr}}\frac{\partial^2}{\partial\phi^2}. \tag{34}$$

The distribution function in ϕ-space $\tilde{f}(\phi) = \int d\boldsymbol{r}\, f(\boldsymbol{r}, \phi)$ obeys the diffusion equation

$$\frac{\partial\tilde{f}}{\partial t} = \frac{1}{\tau_{tr}}\frac{\partial^2\tilde{f}}{\partial\phi^2}. \tag{35}$$

Note that the functional integral (30) is convergent, since the eigenvalues of $\hat{\mathcal{L}}_{\text{eff}}$ are non-negative. In other words, $\hat{\mathcal{L}}_{\text{eff}}$ plays the role of a *finite* regulator in the field-theoretic representation.

In conclusion, having started with a completely reversible kinetic operator with an imaginary spectrum, we ended up with an effective irreversible operator describing a relaxation into the uniform distribution. It is important to emphasize that we did not introduce any collision integral. On the formal level, irreversibility comes from the infinitesimal imaginary part of the spectrum (20). One can draw an analogy between this problem and the problem of Landau damping in a plasma [31]. For the latter, the infinitesimal imaginary part of the spectrum (20) results in a finite damping rate of the *collisionless* plasma. One could say that the initial infinitesimal relaxation rate (which one has to introduce simply to choose the direction of the time evolution) is made finite by the regularization procedure, which for the problem at hand consists of truncating the phase space by eliminating large-$\boldsymbol{k}$ modes. This is quite a general situation: many systems which exhibit irreversible properties such as increase of entropy and relaxation to equilibrium, are actually described by reversible classical mechanics. This phenomenon can be illustrated by the simple example of a gas of classical particles. A proper description must be in terms of the many-particle distribution function $f(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$. The corresponding kinetic operator $\hat{\mathcal{L}}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$ is, of course, fully reversible, that is having a purely imaginary spectrum. In order to obtain a description in terms of the two-particle distribution function $\tilde{f}(\boldsymbol{x}, \boldsymbol{x}')$ one has to truncate the phase space. As a result, the effective operator $\hat{\mathcal{L}}_{\text{eff}}(\boldsymbol{x}, \boldsymbol{x}')$ contains a collision integral, which describes a relaxation into the Maxwell distribution. As shown by this example, there is nothing surprising in the fact that classical dynamics have irreversible properties.

For the diffusion constant (33) to be finite, there has to be a finite correlation length of the magnetic field. For a δ-correlated magnetic field one would have $|\Omega_k|^2 = \text{const.}$, and the integral (33) for $1/\tau_{tr}$ would diverge at large k, making the separation of modes illegitimate. We assume that the case $a = 0$ corresponds to a different physical regime which cannot be described by a diffusive approximation.

Equation (34) is only valid if the distribution in the real space becomes uniform at times much shorter than the diffusion time in the angle space. This is the case if the size of the system L is not too large. In the opposite case of a very large system, $L/v \gg \tau_{tr}$, one has to follow the spirit of the renormalization group approach and divide the large system into smaller sub-systems. In each sub-system, one can separate the modes with $\boldsymbol{k} = 0$ and repeat previous calculations, integrating out the higher modes. The size of a sub-system should be small enough, but still much larger than the correlation length a of the magnetic field. Since each sub-system can be considered as independent, one has to introduce a coordinate $\boldsymbol{R}$ that labels the sub-systems:

$$\psi(\boldsymbol{r}, \phi) = \frac{1}{\sqrt{V}}\Psi(\boldsymbol{R}, \phi) + \frac{1}{\sqrt{V}}\sum_{k\neq 0}\chi_{\boldsymbol{k}}(\boldsymbol{R}, \phi)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \tag{36}$$

where V stands for the volume of a sub-system. This representation effectively separates the degrees of freedom with small and large momenta. Substituting (36) into (27), keeping the terms of up to the second order, and using the fact that the main contribution to the magnetic field comes from large momenta, we obtain

$$S = \int \mathrm{d}\boldsymbol{R}\,\mathrm{d}\phi \left\{\mathrm{i}\omega\Psi^\dagger\Psi - \sum_{k\neq 0}\left[\mathrm{i}v\boldsymbol{k}\cdot\boldsymbol{n}(\phi)\chi_{\boldsymbol{k}}^\dagger\chi_{\boldsymbol{k}} + \frac{1}{\sqrt{V}}\chi_{\boldsymbol{k}}^\dagger\Omega_{\boldsymbol{k}}\frac{\partial\Psi}{\partial\phi} - \frac{1}{\sqrt{V}}\Omega_{\boldsymbol{k}}^*\frac{\partial\Psi^\dagger}{\partial\phi}\chi_{\boldsymbol{k}}\right] - \Psi^\dagger v\boldsymbol{n}(\phi)\cdot\frac{\partial\Psi}{\partial\boldsymbol{R}}\right\}. \tag{37}$$

Integration over $\chi_{\boldsymbol{k}}(\boldsymbol{R}, \phi)$ and $\chi_{\boldsymbol{k}}^*(\boldsymbol{R}, \phi)$ yields

$$S_{\text{eff}} = \int \mathrm{d}\boldsymbol{R}\,\mathrm{d}\phi \left\{\mathrm{i}\omega\Psi^\dagger\Psi - \frac{1}{\tau_{tr}}\frac{\partial\Psi^\dagger}{\partial\phi}\frac{\partial\Psi}{\partial\phi} - \Psi^\dagger v\boldsymbol{n}(\phi)\cdot\frac{\partial\Psi}{\partial\boldsymbol{R}}\right\} \tag{38}$$

which in turn implies a kinetic operator of the form

$$\hat{\mathcal{L}}_{\text{eff}} = -\frac{1}{\tau_{tr}}\frac{\partial^2}{\partial\phi^2} + v\boldsymbol{n}(\phi)\cdot\frac{\partial}{\partial\boldsymbol{R}} \tag{39}$$

with the same transport relaxation time (33).

We are most interested in the spectrum of the effective kinetic operator, which, as explained in the introduction, coincides with the low-lying part of the Perron–Frobenius spectrum. Representing an eigenfunction in the form

$$\Psi(\boldsymbol{R}, \phi) = \psi(\phi)\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{R}} \tag{40}$$

we obtain Mathieu's equation, which is formally equivalent to the Schrödinger equation for a quantum rotator in a uniform electric field,

$$-\frac{1}{\tau_{tr}}\psi'' + \mathrm{i}vq\cos\phi\,\psi = \mathrm{i}\omega\psi. \tag{41}$$

Equation (41) cannot be solved analytically. However, if one is interested in the low-lying excitations, it can be recast in the form suitable for a perturbative treatment:

$$(\hat{H}_0 + \hat{V})\psi = \mathrm{i}\omega\psi \qquad \hat{H}_0 = -\frac{1}{\tau_{tr}}\frac{\mathrm{d}^2}{\mathrm{d}\phi^2} \qquad \hat{V} = \mathrm{i}vq\cos\phi. \tag{42}$$

The unperturbed eigenfunctions and eigenvalues are

$$\psi_m^{(0)} = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}m\phi} \qquad \mathrm{i}\omega_m^{(0)} = \frac{1}{\tau_{tr}}m^2 \qquad m = 0, \pm 1, \pm 2, \ldots. \tag{43}$$

Non-zero transition matrix elements exist only between neighbouring states,

$$V_{m-1,m} = V_{m,m-1} = \frac{\mathrm{i}vq}{2}. \tag{44}$$

Therefore, the first-order corrections are zero. Calculating the second-order correction to the eigenvalue of the ground state ($m = 0$), we find

$$\mathrm{i}\omega(\boldsymbol{q}) = \mathrm{i}\omega_0^{(2)} = \sum_{m(\neq 0)} \frac{V_{0,m}V_{m,0}}{\mathrm{i}\omega_0^{(0)} - \mathrm{i}\omega_m^{(0)}} = Dq^2. \tag{45}$$

The low-lying modes of the effective operator (39) are nothing but diffusion modes in real space. The real space diffusion coefficient D is related to the transport time τ_{tr} by the usual equation (15).

So far, we have shown that the effective Perron–Frobenius operator for the problem at hand contains a term that describes relaxation in the angle space. The effect of this term at large scales is to make the dynamics diffusive. Under the hypothesis of ergodicity (self-averaging), our formula (33) for the inverse scattering time matches the result obtained in the Born approximation after ensemble-averaging, equation (9). The advantage of the field-theoretical approach, however, is that it can be readily generalized to the quantum case, which is the subject of the next section.

3. Quantum dynamics

While the short-time dynamics of a particle in a slowly-varying magnetic field can for the most part be treated classically, the long-time dynamics are influenced strongly by quantum coherence effects which have manifestations both in localization and spectral properties. To study the influence of these coherence effects on the quantum dynamics we will employ the statistical field theory defined in the introduction and applied to the quantum Hamiltonian

$$\hat{H} = \frac{1}{2m}\left(\hat{\boldsymbol{p}} - \frac{e}{c}\boldsymbol{A}\right)^2 \tag{46}$$

where $\boldsymbol{B} = \partial \times \boldsymbol{A}$.

The action, defined by equation (3), retains all the advantages of a simple classical description in the language of the kinetic equation, at the same time accounting for quantum interference effects. However, to apply the quasi-classical theory, we must introduce an appropriate regularization procedure. In doing so, we will take a lesson from the approach employed in the previous section.

Let us first establish a connection between the quantum and the classical field theories, by making an expansion of the action (3) around the saddle point $\mathcal{Q} = \sigma_3^{\mathrm{AR}}$, which is valid at large frequencies $\omega = s\Delta/\hbar$. (In doing so, we make use of symmetry properties of $\mathcal{T}$—see [19].) Applying the parametrization

$$\mathcal{T} = \mathbb{1} + \mathrm{i}\begin{pmatrix} 0 & \bar{\mathcal{B}} \\ \mathcal{B} & 0 \end{pmatrix}_{\mathrm{AR}} \tag{47}$$

expanding to the second order in $\bar{\mathcal{B}}$ and $\mathcal{B}$, and using the antihermiticity of $\hat{\mathcal{L}}$, we recover the classical action

$$S[\bar{\mathcal{B}}, \mathcal{B}] = \hbar\nu \int \mathrm{d}\boldsymbol{r}\, \mathrm{d}\phi\ \mathrm{str}[\bar{\mathcal{B}}(\mathrm{i}\omega - \hat{\mathcal{L}})\mathcal{B}]. \tag{48}$$

At smaller frequencies, higher-order terms in the expansion generate quantum corrections.

Assuming that the timescale at which the motion in real space becomes ergodic is set by the ballistic transport time across the system $\tau_c = L/v$, we conclude that it is the spatially-dependent degrees of freedom that should be integrated out, in the same way as for the classical problem. Quantum effects will come from the interaction between the low-lying relaxation modes of the effective kinetic operator, which is induced by the nonlinearity of the manifold of $\mathcal{Q}$-matrices.

In order to separate slow and fast degrees of freedom in the action (3), instead of the linear decomposition (5) one uses the following nonlinear decomposition,

$$\mathcal{T}(\boldsymbol{x}_\parallel) = \tilde{T}(\boldsymbol{x}_\parallel)T(\boldsymbol{x}_\parallel) \tag{49}$$

where $\tilde{T}(\boldsymbol{x}_\parallel)$ stands for fast fields, $T(\boldsymbol{x}_\parallel)$—for slow fields. It ensures that the $\mathcal{Q}$-matrix always stays on the manifold $\mathcal{Q}^2 = \mathbb{1}$. As explained above, fast fields are spatially-dependent, while slow fields depend only on angle ϕ. Since $\tilde{T}(\boldsymbol{x}_\parallel)$ fluctuates only weakly around $\tilde{T}(\boldsymbol{x}_\parallel) = \mathbb{1}$, we arrive at the representation

$$\mathcal{T}(\boldsymbol{r}, \phi) = \left[\mathbb{1} + \frac{\mathrm{i}}{\sqrt{V}}\sum_{k\neq 0}\begin{pmatrix} 0 & \bar{B}_{-\boldsymbol{k}}(\phi) \\ B_{\boldsymbol{k}}(\phi) & 0 \end{pmatrix}_{\mathrm{AR}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\right] T(\phi) \tag{50}$$

where $T(\phi)$ describes spatially-uniform modes, $B_{\boldsymbol{k}}(\phi)$ and $\bar{B}_{\boldsymbol{k}}(\phi)$ are small fluctuating fields describing the modes with non-zero momenta. Supermatrices $T(\phi)$ and $T^{-1}(\phi)$ parametrize the supermatrix $Q(\phi)$,

$$Q = T^{-1}\sigma_3^{\mathrm{AR}} T \tag{51}$$

so that $Q(\phi)$ obeys the nonlinear constraint $Q^2 = \mathbb{1}$. It is also convenient to introduce the notation

$$M = T\partial_\phi Q T^{-1}. \tag{52}$$

Using the relation $[Q, \partial_\phi Q]_+ = 0$, one can prove the identity

$$2\,\mathrm{str}(M_{\mathrm{AR}}M_{\mathrm{RA}}) = \mathrm{str}(\partial_\phi Q)^2. \tag{53}$$

Now we are prepared to derive an effective field theory for the problem at hand. Substituting the representation (50) into the action (3) with the kinetic operator (22), and making an expansion in B and $\bar{B}$, we obtain

$$S = -\hbar\nu \int \mathrm{d}\phi \sum_{k\neq 0} \mathrm{str}\left[\mathrm{i}v\boldsymbol{k}\cdot\boldsymbol{n}(\phi)\bar{B}_{\boldsymbol{k}}B_{\boldsymbol{k}} + \frac{\mathrm{i}}{2}\bar{B}_{\boldsymbol{k}}\Omega_{\boldsymbol{k}}M_{\mathrm{AR}} + \frac{\mathrm{i}}{2}\Omega^*_{\boldsymbol{k}}M_{\mathrm{RA}}B_{\boldsymbol{k}}\right] \\ -\frac{\mathrm{i}\hbar\omega}{4\Delta}\int \mathrm{d}\phi\ \mathrm{str}\,\sigma_3^{\mathrm{AR}}Q. \tag{54}$$

Integration over the fast degrees of freedom yields the effective action

$$S_{\mathrm{eff}} = -\frac{\hbar}{8\Delta}\int \mathrm{d}\phi\ \mathrm{str}\left[\frac{1}{\tau_{tr}}(\partial_\phi Q)^2 + 2\mathrm{i}\omega\sigma_3^{\mathrm{AR}}Q\right] \tag{55}$$

which has the form of a one-dimensional diffusive σ-model, the diffusion taking place in the angle space. The transport time is given by the same formula (33) as for the classical problem. To arrive at equation (55), we have used the identity (53).

If $\omega\tau_{tr} \gg 1$, the main contribution to the functional integral comes from around the saddle point $Q = \sigma_3^{AR}$. In the opposite case, $\omega\tau_{tr} \ll 1$, one must take into account the whole supermatrix manifold. Spectral correlation functions are then determined by universal Wigner–Dyson statistics. Corrections to this universal behaviour can be calculated by performing an expansion in $1/g$ analogous to that employed by Kravtsov and Mirlin [32], where

$$g = \frac{\hbar}{\Delta\tau_{tr}} \tag{56}$$

plays the role of dimensionless conductance.

At larger scales the low-lying modes of the Perron–Frobenius operator acquire a spatial dependence. This can be accounted for by using another representation, instead of (50), that separates degrees of freedom with small and large momenta (compare with the previous section). The effective action takes the form of the ballistic σ-model,

$$S_{\text{eff}} = -\frac{\hbar\nu}{4}\int \mathrm{d}\boldsymbol{r}\,\mathrm{d}\phi\ \mathrm{str}\left[\frac{1}{2\tau_{tr}}(\partial_\phi Q)^2 + \mathrm{i}\omega\sigma_3^{AR}Q - 2T^{-1}\sigma_3^{AR}v\boldsymbol{n}(\phi)\cdot\nabla T\right]. \tag{57}$$

Note that a similar type of action was proposed by Muzykantskii and Khmelnitskii [2], where a collision integral was introduced as a phenomenological term to describe quantum scattering by a δ-correlated impurity potential. The second term in the action (57) describes the diffusion in the angle space and effectively plays the role of a collision integral in the sense that it is responsible for relaxation of the momentum-dependent degrees of freedom. However, its origin in our case is quite different, since it appears as a result of the regularization of the field theory.

Finally, one can establish a relation between the ballistic σ-model (57) and the conventional diffusive σ-model using the procedure similar to that of [2]. Anticipating a rapid relaxation in the momentum space, and a slow variation of the spatial modes, one can integrate out the momentum-dependent degrees of freedom to obtain the action (1) with the diffusion constant (15). Therefore, the transport properties of the system at very large scales are dominated by the localization effects.

To summarize, in this paper we applied the recently-developed field theory of quantum chaos to the problem of a particle propagating in a non-uniform magnetic field. We proposed a new approach to the regularization of the ballistic σ-model (3), and obtained an effective description of spectral and transport properties of the quantum system in terms of the low-lying modes of the *irreversible* dynamics of its classical counterpart. We argued that this method is quite general and can be applied to a variety of chaotic systems.

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References

[1] Gutzwiller M C 1990 *Chaos in Classical and Quantum Mechanics* (New York: Springer)
[2] Muzykantskii B A and Khmelnitskii D E 1995 *JETP Lett.* **62** 76
[3] Geim A, Bending S and Grigorieva I 1992 *Phys. Rev. Lett.* **69** 2252
[4] Smith A *et al* 1994 *Phys. Rev.* B **50** 14 726
[5] Mancoff F B *et al* 1995 *Phys. Rev.* B **51** 13 269
[6] Ioffe L B and Larkin A I 1989 *Phys. Rev.* B **39** 8988
[7] Nagaosa N and Lee P A 1990 *Phys. Rev. Lett.* **64** 2450
[8] Halperin B I, Lee P A and Read N 1993 *Phys. Rev.* B **47** 7312
[9] Kalmeyer V and Zhang S-C 1992 *Phys. Rev.* B **46** 9889

[10] Hedegård P and Smith A 1995 *Phys. Rev.* B **51** 10 869
[11] Mirlin A D, Polyakov D G and Wölfle P 1998 *Phys. Rev. Lett.* **80** 2429
[12] Khveshchenko D V 1996 *Phys. Rev. Lett.* **77** 1817
[13] Isichenko M B 1992 *Rev. Mod. Phys.* **64** 961
[14] Zielinski L, Chaltikian K, Birnbaum K, Marcus C M, Campman K and Gossard A C 1998 *Europhys. Lett.* **42** 73
[15] Miller J and Wang J 1996 *Phys. Rev. Lett.* **76** 1461
[16] Aronov A G, Mirlin A D and Wölfle P 1994 *Phys. Rev.* B **49** 16 609
[17] Altshuler E, Aronov A G, Mirlin A D and Wölfle P 1995 *Europhys. Lett.* **29** 239
[18] Efetov K B 1983 *Adv. Phys.* **32** 53
[19] Simons B D, Agam O and Andreev A V 1997 *J. Math. Phys.* **38** 1982
[20] Bohigas O, Giannoni M J and Schmidt C 1984 *Phys. Rev. Lett.* **52** 1
[21] Andreev A V, Simons B D, Agam O and Altshuler B L 1996 *Nucl. Phys.* B **482** 536
[22] Zirnbauer M R 1999 *Supersymmetry and Trace Formulae: Chaos and Disorder* ed I V Lerner, J P Keating and D E Khmelnitskii (Oxford: Plenum) pp 153–72
[23] Blanter Ya M, Mirlin A D and Muzykantskii B A 1998 *Phys. Rev. Lett.* **80** 4161
[24] Tripathi V and Khmelnitskii D E 1998 *Phys. Rev.* B **58** 1122
[25] Gaspard P, Nicolis G, Provata A and Tasaki S 1995 *Phys. Rev.* E **51** 74
[26] Ruelle D 1978 *Statistical Mechanics, Thermodynamic Formalism* (Reading, MA: Addison-Wesley)
[27] Nicolis G and Nicolis C 1988 *Phys. Rev.* A **38** 427
[28] Hasegawa H H and Driebe D J 1994 *Phys. Rev.* E **50** 1781
[29] Ruelle D 1986 *Phys. Rev. Lett.* **56** 405
[30] Pollicot M 1990 *Ann. Math.* **131** 331
[31] Lifshitz E M and Pitaevskii L P 1981 *Physical Kinetics* (Oxford: Pergamon)
[32] Kravtsov V E and Mirlin A D 1994 *JETP Lett.* **60** 656